DEGREE COMPLEXITY OF A FAMILY OF BIRATIONAL MAPS: II. EXCEPTIONAL CASES

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Abstract. We compute the degree complexity of the family of birational maps considered in [2] for all exceptional cases. Some interesting properties of the family are also given.

1. Introduction

We continue the work of [2] where we considered a family k_F of birational maps of the plane determined by a choice of polynomial $F(w) = a_0 + a_1 w + \ldots + a_n w^n$ (the definition of the family k_F will be recalled in Section 2). In [2] we determined the degree complexity $\delta(k_F)$ in the generic case. If $\delta(k_{\widehat{F}})$ is less than the generic value, then we say that $\widehat{F}$ is exceptional. (The set of exceptional parameters is a nowhere dense algebraic subset.) This corresponds to cases of degree reduction which are especially interesting because they correspond to the maps that have special symmetries.

As seen in [2], there is a fundamental difference between the cases where n, the degree of F, is even or odd.

The complexity degree $\delta(k_F)$ in the case n is even is given by

Theorem 1. *Suppose that $n = deg(F)$ is even.*

Case 1: If $a_0 \neq \frac{2}{1+m}$ for all integers $m \geq 0$ then $\delta(k_F)$ is the largest root of $x^2 - (n+1)x - 1$.

Case 2: If $a_0 = \frac{2}{1+m}$ for some integer $m \geq 0$ then $\delta(k_F)$ is the largest real root of the polynomial $x^{2m+1}(x^2 - (n+1)x - 1) + x^2 + n$.

The complexity degree $\delta(k_F)$ in the case n is odd is given by

Theorem 2. *Suppose that $n = deg(F) \geq 3$ is odd. Let linear functions $L_j : \mathbb{C}^{n+1} \to \mathbb{C}$ (for $0 \leq j \leq n$) be defined as in (5.1)*

$$L_j(a_0, a_1, \ldots, a_n) = -(a_{n-j} + a_{n-j+1}) - [-a_n \binom{n}{j} + a_{n-1}\binom{n-1}{j-1} + \ldots + (-1)^{j+1} a_{n-j}\binom{n-j}{0}],$$

where

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}$$

with $n! = n(n-1)\ldots 2.1$ the factorial of n.

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Let $0 \leq h \leq n-2$ be the largest integer in $[0, n-2]$ for which

$$L_j(a_0, a_1, \ldots, a_n) = 0$$

for all $0 \leq j \leq h$.

Case 1: $h < n-2$, and $a_0 \neq 2/(1+m)$ for all integers $m \geq 0$. Then $\delta(k_F)$ is the largest real root of the polynomial $x^3 - nx^2 - (n+1-h)x - 1$.

Case 2: $h < n-2$, and $a_0 = 2/(1+m)$ for some integer $m \geq 0$. Then $\delta(k_F)$ is the largest real root of the polynomial $x^{2m+1}(x^3 - nx^2 - (n-h+1)x - 1) + x^3 + x^2 + nx + n - h - 1$.

Case 3: $h = n-2$, and $a_0 \neq 2/(1+m)$ for all integers $m \geq 0$, and $a_0 \neq \frac{n+1}{2} + \frac{l}{2(1+l)}$ for all integers $l \geq 0$. Then $\delta(k_F)$ is the largest real root of the polynomial $x^3 - nx^2 - 2x - 1$.

Case 4: $h = n-2$, and $a_0 = 2/(1+m)$ for some integer $m \geq 0$, and $a_0 \neq \frac{n+1}{2} + \frac{l}{2(1+l)}$ for all integers $l \geq 0$. Then $\delta(k_F)$ is the largest real root of the polynomial $x^{2m}(x^3 - nx^2 - 2x - 1) + x^2 + x + n$.

Case 5: $h = n-2$, and $a_0 \neq 2/(1+m)$ for all integers $m \geq 0$, and $a_0 = \frac{n+1}{2} + \frac{l}{2(1+l)}$ for some integer $l \geq 0$. Then $\delta(k_F)$ is the largest real root of the polynomial $x^{2l+2}(x^3 - nx^2 - 2x - 1) + nx^2 + x + 1$.

Case 6: $h = n-2$, and $a_0 = 2/(1+m)$ for some integer $m \geq 0$, and $a_0 = \frac{n+1}{2} + \frac{l}{2(1+l)}$ for some integer $l \geq 0$. Then $n = 3$, $a_0 = 2$, and the map k_F is exactly the family considered in Section 5 in [2]. Hence in this case $\delta(k_F) = 1$.

There are two interesting phenomena which occur to the maps k_F:

1. The first phenomenon, which occurs when $n \geq 3$ is odd, is what we call "double point-blowups". This means that in Theorem 2, if $h < n-2$ then h is an even number, while if $h = n-2$ then $L_{n-1}(a_0, \ldots, a_n) = 0$. We will give an example exploring the case $n = 3$ in Section 4 to illustrate this phenomenon. This is a consequence of the results about a system of linear equations that we will explore in Section 5.

2. The other phenomenon is that there is no automorphism if $n = deg(F)$ is different from 1 or 3. This is also a sequence of the results about a system of linear equations that we mentioned above. The exact formulation of this phenomenon is

Theorem 3. *Let $n = deg(F)$. If $n \neq 1$ and $n \neq 3$ then there is no space Z which satisfies the following two conditions:*

1) Z is constructed from $\mathbb{P}^2$ by a finite number of point blowing-ups.

2) The induced map $k_Z : Z \to Z$ is an automorphism.

If $n = 3$ then a space Z with properties 1) and 2) exists iff $F(z) = a_3 z^3 + a_3 z^2 + a_1 z + 2$.

Proof. We consider three cases:

Case 1: $n = deg(F)$ is even. Then from the proof of Theorem 1, it follows that we can not resolve the point $\frac{1}{a_n} \in P_{n-1}$, which is the image of some exceptional curves, to obtain an automorphism.

Case 2: $n = deg(F) \geq 5$ is odd. Then it follows from the proof of Theorem 4, we can have an automorphism iff simultaneously $a_0 = \frac{2}{m+1}$ for some $m = 0, 1, 2, \ldots$, and $a_0 = \frac{n+1}{2} + \frac{l}{2(l+1)}$ for some $l = 0, 1, 2, \ldots$. If $a_0 = \frac{2}{m+1}$ then obviously $a_0 \leq 2$, while if $a_0 = \frac{n+1}{2} + \frac{l}{2(l+1)}$ and $n \geq 5$ then $a_0 \geq 3$. Hence in this case we do not have an automorphism.

Case 3: $n = deg(F) = 3$. Then use Lemma 2 we have that a space Z with properties 1) and 2) in the statement of Theorem 3 exists iff $F(z) = a_3z^3 + a_3z^2 + a_1z + 2$. □

Theorem 3 shows that the family k_F corresponding with the maps $F(z) = az^3 + az^2 + b + 2$ (with $a \neq 0$) as described in Section 5 in [2] is the only family of automorphism in the whole family k_F, besides the case $n = deg(F) = 1$ which was known previously (see [3] and [1]).

2. Properties of k_F

We review in this Section some results in [2].

Let $\mathbb{P}^2$ be the complex projective space of dimension 2, with coordinate $[x_0 : x_1 : x_2]$. Given a polynomial of degree n

$$F(z) = a_nz^n + a_{n-1}z^{n-1} + \ldots + a_1z + a_0$$

where $a_n \neq 0$, we define a birational map $k = k_F : \mathbb{P}^2 \to \mathbb{P}^2$ (see [2]) by the formula $k_F = j_F \circ i$ where j_F and i are involutions defined in the open Zariski dense set $\mathbb{C}^2$ of $\mathbb{P}^2$ by

$$j_F(x,y) = (-x + F(y), y),\ i(x,y) = (1 - x - \frac{x-1}{y}, -y - 1 - \frac{y}{x-1}).$$

The map $k_F = [k_0 : k_1 : k_2]$ is given in homogeneous coordinates as

$$\begin{aligned} k_0 &= (x_0x_1 - x_0^2)^n x_2, \\ k_1 &= x_0^{n-1}(x_0 - x_1)^{n+1}(x_0 + x_2) + x_2 \sum_{j=0}^{n} a_j (x_0x_1 - x_0^2)^{n-j}(x_2^2 - x_0x_1 - x_1x_2)^j, \\ k_2 &= x_2(x_0x_1 - x_0^2)^{n-1}(x_2^2 - x_0x_1 - x_1x_2). \end{aligned}$$

It is worth to write out also the non-homogeneous form of k which is convenient in computation

(2.1)

$$k[x_0 : x_1 : x_2] = [1 : \frac{(x_1 - x_0)(x_2 + x_0)}{x_0x_2} + F(-1 - \frac{x_1x_2}{x_0(x_1 - x_0)}) : -1 - \frac{x_1x_2}{x_0(x_1 - x_0)}]$$

The inverse map is

(2.2)

$$k^{-1}[x_0 : x_1 : x_2] = [1 : (1 + \frac{x_1}{x_0} - F(\frac{x_2}{x_0}))(1 + \frac{x_0}{x_2}) : -1 - \frac{x_2}{x_0} + \frac{x_2}{x_0 + x_1 - x_0F(x_2/x_0)}].$$

We recall the following notations from [2]:

$$\begin{aligned} C_1 &= \{x_0 = 0\},\ C_2 = \{x_0 = x_1\},\ C_3 = \{x_2 = 0\}, C_4 = \{-x_0^2 + x_0x_1 + x_1x_2 = 0\}, \\ C_1' &= C_1,\ C_2' = \{1 + \frac{x_1}{x_0} - F(\frac{x_2}{x_0}) = 0\},\ C_3' = C_3, \\ C_4' &= \{\frac{x_2}{x_0} - (1 + \frac{x_2}{x_0})(1 + \frac{x_1}{x_0} - F(\frac{x_2}{x_0})) = 0\}. \end{aligned}$$

The exceptional hypersurfaces of k_F are mapped as

$$k_F :\ C_4 \mapsto [1 : -1 + a_0 : 0] \in C_3,\ C_1 \cup C_2 \cup C_3 \mapsto e_1.$$

The points of indeterminate of k_F are $e_1 = [0 : 1 : 0], e_2 = [0 : 0 : 1]$, and $e_{01} = [1 : 1 : 0]$. The exceptional curves for k_F^{-1} are mapped as

$$k_F^{-1} :\ C_1' \cup C_3' \mapsto e_1,\ C_2' \mapsto e_2,\ C_4' \mapsto e_{01}.$$

Notation: Let Z be a complex manifold and let $k_Z : Z \to Z$ be a birational map. Recall that $k_Z : Z \to Z$ is (1,1)-regular or algebraically stable (or A.S. for brevity) if it satisfies

$$(k_Z^p)^* = (k_Z^*)^p \tag{2.3}$$

for all $p \in \mathbb{N}$, where $k_Z^* : Pic(Z) \to Pic(Z)$ is the induced pull-back of k_Z on the Picard group of Z, and similarly for $(k_Z^p)^*$. In [3], it was proved that any birational map of a compact Kahler surface can be (1,1)-regularized after a finite number of point-blowups.

3. Proof of Theorem 1

First we recall construction of the space X constructed in Section 3 of [2]: Define a complex manifold $\pi_X : X \to \mathbb{P}^2$ (see Figure 3.1 in [2]) by blowing ups points $e_1, p_1, \ldots, p_{n-1}$ in the following order:

i) blowup $e_1 = [0 : 1 : 0]$ and let E_1 denote the exceptional fiber over e_1,

ii) blowup $q = E_1 \cap C_4$ and let Q denote the exceptional fiber over q,

iii) blowup $p_1 = E_1 \cap C_1$ and let P_1 denote the exceptional fiber over e_1,

iv) blowup $p_j = P_j \cap E_1$ with exceptional fiber P_j for $2 \leq j \leq n-1$.

The exceptional curves of the induced map k_X are $C_1, C_2, C_4, P_1, \ldots, P_{n-2}$. All the curves $C_1, C_2, P_1 \ldots, P_{n-2}$ are mapped to the same point $1/a_n \in P_{n-1}$, while C_4 is mapped to the point $[1 : -1 + a_0 : 0] \in C_3$. By Lemmas 3.2 and 3.3 in [2], the only way that an exceptional curve can be mapped to a point of indeterminacy is if $a_0 = 2/(m+1)$ for some $m \in \mathbb{N}$, and in this case we have $k_X^{2m+1} C_4 = [1 : 1 : 0]$.

If $a_0 = 2/(m+1)$ we construct the new manifold Z by blowing up the manifold X at the points

$$\begin{aligned} r_0 &= [1 : -1 + a_0 : 0] \in C_3, \\ q_1 &= k_X(r_0) \in Q,\ r_1 = k_X(q_1) \in C_3, \\ &\ldots \\ q_m &= k_X(r_{m-1}) \in Q,\ r_m = k_X(q_m) = [1 : 1 : 0] \in C_3. \end{aligned}$$

Call $R_0, Q_1, R_1, \ldots, Q_m, R_m$ the exceptional fibers of this blowup.

Lemma 1. *If $a_0 = 2/(m+1)$ and Z is constructed as above then the curves $C_4, R_0, Q_1, R_1, \ldots, Q_m, R_m$ are not exceptional for k_Y.*

Proof. It suffices to check that C_4 is not exceptional. We choose a local projection for R_0 as

$$Z \ni (s, u) \mapsto [1 : -1 + a_0 + su : s].$$

In this coordinate chart $R_0 = \{s = 0\}$. If we rewrite $k[x_0 : x_1 : x_2]$ as

$$k[x_0 : x_1 : x_2] = [1 : -1 - \frac{x_0^2 - x_0x_1 - x_1x_2}{x_0x_1} + F(\frac{x_0^2 - x_0x_1 - x_1x_2}{x_0(x_1 - x_0)}) : \frac{x_0^2 - x_0x_1 - x_1x_2}{x_0(x_1 - x_0)}]$$

then it can be seen that

$$k_Z : C_4 \ni [x_0 : x_1 : x_2] \mapsto -1 + \frac{x_0}{x_1} \in R_0.$$

Hence C_3 is not exceptional. $\square$

It follows (see [3]) that k_Z is A.S. Thus we obtain $\delta(k_F)$ as the spectral radius of k_Z^*. Now we compute k_Z^*.

For brevity, we denote by E_1 the strict transform $\widetilde{E_1}$ in Z of the exceptional fiber E_1, and the same notation E_1 is also used for the class in $Pic(Z)$ of $\widetilde{E_1}$. The same convenience is applied to $C_1, C_2, C_3, P_1, \ldots, P_{n-1}, Q, Q_1, \ldots, Q_m, R_0, \ldots, R_m$. Let H_Z denote the class in $Pic(Z)$ of the strict transform of a generic line H in $\mathbb{P}^2$. Then $H_Z, E_1, P_1, \ldots, P_{n-1}, Q, Q_1, \ldots, Q_m, R_0, \ldots, R_m$ form a basis for the space $Pic(Z)$. C_1, C_2, C_3, C_4 can be represented in this basis as

$$\begin{aligned}
C_1 &= H_Z - E_1 - Q - \sum_{j=1}^{n-1}(j+1)P_j - \sum_{j=1}^{m} Q_j, \\
C_2 &= H_Z - R_m, \\
C_3 &= H_Z - E_1 - Q - \sum_{j=1}^{n-1} jP_j - \sum_{j=1}^{m} Q_j - \sum_{j=0}^{m} R_j, \\
C_4 &= 2H_Z - E_1 - 2Q - \sum_{j=1}^{n-1} jP_j - 2\sum_{j=1}^{m} Q_j - R_m.
\end{aligned}$$

The induced map k_Z acts as follows

$$\begin{aligned}
&k_Z : E_1 \mapsto E_1,\ P_{n-1} \mapsto P_{n-1},\ C_1, C_2, P_1, \ldots, P_{n-2} \mapsto \frac{1}{a_n} \in P_{n-1},\ Q \mapsto C_3 \mapsto Q, \\
&k_Z : C_4 \mapsto R_0 \mapsto Q_1 \mapsto R_1 \mapsto \ldots \mapsto Q_m \mapsto R_m \mapsto C_4', \\
&k_Z^{-1} : C_1, P_1, \ldots, P_{n-1} \mapsto -\frac{1}{a_n} \in P_{n-1}.
\end{aligned}$$

From this, the induced map $k_Z^* : Pic(Z) \to Pic(Z)$ is as follows

$$\begin{aligned}
k_Z^*(H_Z) &= (2n+1)H_Z - nE_1 - (n+1)Q - (n+1)\sum_{j=1}^{n-1} jP_j - (n+1)\sum_{j=1}^{m} Q_j - (n+1)R_m, \\
k_Z^*(E_1) &= E_1, \\
k_Z^*(Q) &= C_3 = H_Z - E_1 - Q - \sum_{j=1}^{n-1} jP_j - \sum_{j=1}^{m} Q_j - \sum_{j=0}^{m} R_j, \\
k_Z^*(P_j) &= 0,\ 1 \le j \le n-2, \\
k_Z^*(P_{n-1}) &= C_1 + C_2 + \sum_{j=1}^{n-1} P_j = 2H_Z - E_1 - Q - \sum_{j=1}^{n-1} jP_j - \sum_{j=1}^{m} Q_j - R_m, \\
k_Z^*(R_0) &= C_4 = 2H_Z - E_1 - 2Q - \sum_{j=1}^{n-1} jP_j - 2\sum_{j=1}^{m} Q_j - R_m, \\
k_Z^*(R_j) &= Q_j,\ 1 \le j \le m,\ k_Z^*(Q_j) = R_{j-1},\ 1 \le j \le m.
\end{aligned}$$

From the above we find that the characteristic polynomial of k_Z^* is

$$P(x) = -x[x^{2m+1}(x^2 - (n+1)x - 1) + x^2 + n].$$

From this, Theorem 1 follows.

4. Example: case $n = 3$

In this section we explore the map k when $n = deg(F) = 3$. In this case $F(z) = a_3z^3 + a_2z^2 + a_1z + a_0$ where $a_3 \neq 0$.

Let $Y_1 = Y$ be the manifold and E, Q, P_1, P_2, P_3 the exceptional fibers constructed in Section 4 in [2]. The action of the induced map $k_{Y_1} : Y_1 \to Y_1$ is

$$C_1, C_2 \overset{k_{Y_1}}{\mapsto} -\frac{a_2}{a_3^2} \in P_3,\ C_1 \overset{k_{Y_1}^{-1}}{\mapsto} \frac{-2a_3 + a_2}{a_3^2} \in P_3,$$

and

$$k_{Y_1} : P_1 \ni u \mapsto -\frac{(1 + a_2u)}{a_3^2u} \in P_3,\ P_3 \ni u \mapsto \frac{1}{-2a_3 + a_2 - a_3^2u} \in P_1.$$

(In these formula, we use the same local coordinates as that of [2].)

1) Case 1: $a_2 \neq a_3$. Then the orbit of exceptional curves C_1, C_2 will never land on an indeterminacy point. Hence depending on whether $a_0 = \frac{2}{m+1}$ for some $m = 0, 1, 2, \ldots$ or not, we will decide to perform the blowups as in the proof of Theorem 1 or not. For the resulting manifold Z, the induced map k_Z is A.S.

2) Case 2: $a_2 = a_3$. Then the map k_{Y_1} is not A.S because

$$\frac{-2a_3 + a_2}{a_3^2} = -\frac{a_2}{a_3^2} = -\frac{1}{a_3},$$

that is the point $-\frac{1}{a_3} \in P_3$ is both an indeterminate point and the image of exceptional curves C_1, C_2. We blowup the space Y_1 at the point $-\frac{1}{a_3} \in P_3$. Call Y_2 the resulting manifold and P_4 the exceptional fiber of this blowup. We choose a coordinate projection for P_4 as

$$Y_2 \ni (s, u) \mapsto [s^3(\frac{1}{a_3} - \frac{s}{a_3} + s^2u) : 1 : s^2(\frac{1}{a_3} - \frac{s}{a_3} + s^2u)].$$

Recall that this means that in this local coordinate P_4 is given by the equation $s = 0$.

Then the action of the induced map $k_{Y_2} : Y_2 \to Y_2$ is

$$\begin{aligned} k_{Y_2} :\ P_4 \ni u &\mapsto [0 : 1 : \frac{1}{-a_3 + a_1 + a_3^2u}] \in C_1, \\ k_{Y_2} :\ C_1 \ni [0 : 1 : u] &\mapsto \frac{1 + a_3u - a_1u}{a_3u} \in P_4, \end{aligned}$$

and

$$k_{Y_2} :\ C_2 \mapsto [\frac{a_3 - a_1}{a_3^2}]_{P_4} \mapsto [0 : 0 : 1] = e_2,$$

Since e_2 is an indeterminate point, it follows that k_{Y_2} is not A.S., and we need to blowup more times. This is what we called "double point-blowups" in Section 1.

We blowup Y_2 at points $\frac{a_3 - a_1}{a_3^2} \in P_4$ and e_2. Call Y_3 the resulting manifold and P_5, E_2 the exceptional fibers of this blowup.

We choose a coordinate projection for P_5 as

$$Y_3 \ni (s, u) \mapsto [s^3(\frac{1}{a_3} - \frac{s}{a_3} + \frac{a_3 - a_1}{a_3^2}s^2 + s^3u) : 1 : s^2(\frac{1}{a_3} - \frac{s}{a_3} + \frac{a_3 - a_1}{a_3^2}s^2 + s^3u)],$$

and a coordinate projection for E_2 as

$$Y_3 \ni (s, u) \mapsto [s : su : 1].$$

The action of the induced map k_{Y_3} is

$$\begin{aligned} k_{Y_3}:\ P_5 \ni u &\mapsto -a_3^2 u - a_3 + 2a_1 + a_0 - 4 \in E_2, \\ k_{Y_3}: E_2 \ni u &\mapsto \frac{u + a_3 - 2a_1 + a_0 - 1}{a_3^2} \in P_5, \\ k_{Y_3}^2: E_2 \ni u &\mapsto u + 2a_0 - 5 \in E_2, \end{aligned}$$

and

$$k_{Y_3}:\ C_2 \mapsto -\frac{a_3 - 2a_1 + a_0}{a_3^2} \in P_5 \mapsto 2a_0 - 4 \in E_2.$$

This map has only one more indeterminate point at $0 \in E_2$.

2.1) Subcase 2.1: $a_0 \neq 2 + \frac{l}{2(l+1)}$ for any $l = 0, 1, 2, \ldots$. Then the orbit of the exceptional curve C_2 will never land on an indeterminacy point. Hence depending on whether $a_0 = \frac{2}{m+1}$ for some $m = 0, 1, 2, \ldots$ or not, we will decide to perform the blowups as in Theorem 1 or not. For the resulting manifold Z, the induced map k_Z is A.S.

2.2) Subcase 2.2: $a_0 = 2 + \frac{l}{2(l+1)}$ for some $l = 0, 1, 2, \ldots$. In this case we do a series of blowups at the point $0 \in E_2$ and a finite number of its previous images in the same way as we did in Theorem 1. If also $a_0 = \frac{2}{m+1}$ for some $m = 0, 1, 2, \ldots$ we also perform the series of blowups in Theorem 1. For the resulting space Z, the induced map $k_Z : Z \to Z$ is A.S.

Note that if both $a_0 = 2 + \frac{l}{2(l+1)}$ for some $l = 0, 1, 2, \ldots$ and $a_0 = \frac{2}{m+1}$ for some $m = 0, 1, 2, \ldots$ then $a_0 = 2$. In particular, from the above analysis, we get that

Lemma 2. *If $n = 3$, then a space Z satisfying 1) and 2) of Theorem 3 exists iff $F(z) = a_3 z^3 + a_3 z^2 + a_1 z + 2$ where $a_3 \neq 0$.*

5. A SYSTEM OF LINEAR EQUATIONS

In this section we explore a system of linear equations which is related to the map k_F.

Fixed $n \in \mathbb{N}$, where n is not necessarily odd. We define linear functions $L_j : \mathbb{C}^{n+1} \to \mathbb{C}$ as follows

(5.1)

$$L_j(a_0, a_1, \ldots, a_n) = -(a_{n-j} + a_{n-j+1}) - [-a_n \binom{n}{j} + a_{n-1}\binom{n-1}{j-1} + \ldots + (-1)^{j+1} a_{n-j}\binom{n-j}{0}],$$

for $0 \leq j \leq n$, and where

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}$$

with $n! = n(n-1)\ldots 2.1$ the factorial of n. Functions L_j for some first values of j are:

$$\begin{aligned} L_0 &= -a_n - [-a_n] = 0, \\ L_1 &= -(a_n + a_{n-1}) - [-na_n + a_{n-1}] = (n-1)a_n - 2a_{n-1}, \\ L_2 &= -(a_{n-1} + a_{n-2}) - [-a_n\binom{n}{2} + a_{n-1}\binom{n-1}{1} - a_{n-2}\binom{n-2}{0}] = \frac{n}{2} L_1. \end{aligned}$$

We will explore the properties of systems of linear equations of the form

$$L_j(a_0, a_1, \ldots, a_n) = 0 \tag{5.2}$$

for all $j = 0, 1, 2, \ldots, m$, where $0 \leq m < n$ is a constant integer. It will be convenient to write equations (5.2) as

$$(5.3)\quad -(a_{n-j} + a_{n-j+1}) = -a_n \binom{n}{j} + a_{n-1}\binom{n-1}{j-1} + \ldots + (-1)^{j+1} a_{n-j}\binom{n-j}{0}$$

Changing the order of indexes, the equations (5.3) can be written in a more convenient form

$$(5.4)\qquad -(b_j + b_{j-1}) = -b_0\binom{n}{j} + b_1\binom{n-1}{j-1} + \ldots + (-1)^{j+1} b_j\binom{n-j}{0}.$$

The following results will be used to prove the phenomenon "double point-blowups" that we mentioned in Section 1.

Lemma 3. *If $0 \leq m < n$, and m is odd, and if $b_0, b_1, \ldots, b_n$ satisfy the equations (5.4) for all $j = 1, 3, 5, \ldots, m$ then $b_0, b_1, \ldots, b_n$ also satisfy (5.4) for all $j = 0, 2, 4, \ldots, m+1$.*

Proof. Fixed $0 \leq m < n$, where m is odd. Let $b_0, b_1, \ldots, b_n$ satisfy the equations (5.4) for all $j = 1, 3, 5, \ldots, m$. To prove Lemma 3 it suffices to prove the following claim:

Claim 1: $b_0, b_1, \ldots, b_n$ also satisfy (5.4) for $j = m+1$.

The proof is divided in several steps.

i) Reduction 1: In equations (5.4) with $j = 1, 3, \ldots, m$, pushing all b_i with i odd to the left hand-sided and pushing all b_i with i even to the right hand-sided we can rewrite them as

$$\begin{aligned}
&2b_1 = b_0\binom{n-1}{1},\\
&b_1\binom{n-1}{2} + 2b_3 = b_0\binom{n}{3} + b_2\binom{n-3}{1},\\
&b_1\binom{n-1}{4} + b_3\binom{n-3}{2} + 2b_5 = b_0\binom{n}{5} + b_2\binom{n-2}{3} + b_4\binom{n-5}{1},\\
&\vdots\\
&b_1\binom{n-1}{m-1} + b_3\binom{n-3}{m-3} + \ldots + b_{m-2}\binom{n-m+2}{2} + 2b_m\\
&= b_0\binom{n}{m} + b_2\binom{n-2}{m-2} + \ldots + b_{m-3}\binom{n-m+3}{3} + b_{m-1}\binom{n-m}{1}.
\end{aligned}$$

The equation (5.4) for $j = m+1$ which we want to prove in Claim 1 can be written as

$$\begin{aligned}
&b_1\binom{n-1}{m} + b_3\binom{n-3}{m-2} + \ldots + b_{m-2}\binom{n-m+2}{3} + b_m\binom{n-m+1}{1}\\
= \;&b_0\binom{n}{m+1} + b_2\binom{n-2}{m-1} + \ldots + b_{m-1}\binom{n-m+1}{2}.
\end{aligned}$$

ii) Reduction 2: For any value of $b_0, b_2, b_4, \ldots, b_{m-1}$ there exists a unique solution $b_1, b_3, \ldots, b_m$ to the system (5.4) for $j = 1, 3, \ldots, m$. For a proof of this claim we can use the rewritten system in Reduction 1.

iii) Reduction 3: Claim 1 is true in general case if we can prove it is true for the special case $b_0 = 1, b_2 = b_4, \ldots = 0$. For a proof use the special structure of the rewritten system in Reduction 1.

From now on in this proof we will assume that $b_0 = 1, b_2 = b_4 = \ldots = 0$. We rewrite Reduction 1 as

iv) Reduction 4: In equations (5.4) with $j = 1, 3, \ldots, m$, pushing all b_i with i odd to the left hand-sided and pushing all b_i with i even to the right hand-sided we can rewrite them as

$$\begin{aligned}
2b_1 &= \binom{n-1}{1}, \\
b_1\binom{n-1}{2} + 2b_3 &= \binom{n}{3}, \\
b_1\binom{n-1}{4} + b_3\binom{n-3}{2} + 2b_5 &= \binom{n}{5}, \\
&\vdots \\
b_1\binom{n-1}{m-1} + b_3\binom{n-3}{m-3} + \ldots + b_{m-2}\binom{n-m+2}{2} + 2b_m &= \binom{n}{m}.
\end{aligned}$$

The equation (5.4) for $j = m+1$ which we want to prove in Claim 1 can be written as

$$b_1\binom{n-1}{m} + b_3\binom{n-3}{m-2} + \ldots + b_{m-2}\binom{n-m+2}{3} + b_m\binom{n-m+1}{1} = \binom{n}{m+1}.$$

v) Reduction 5: Define

$$\begin{aligned}
\beta_1 &= \frac{b_1}{n}, \\
\beta_3 &= \frac{b_3}{n(n-1)(n-2)}, \\
\beta_5 &= \frac{b_5}{n(n-1)(n-2)(n-3)(n-4)}, \\
&\cdots
\end{aligned}$$

then $\beta_!, \beta_3, \beta_5, \ldots$ satisfy the following system of equations

$$\begin{aligned}
2\beta_1 &= 1 - \frac{1}{n}, \\
\frac{\beta_1}{2!} + 2\beta_3 &= \frac{1}{3!}, \\
\frac{\beta_1}{4!} + \frac{\beta_3}{2!} + 2\beta_5 &= \frac{1}{5!}, \\
&\ldots, \\
\frac{\beta_1}{(m-1)!} + \frac{\beta_3}{(m-3)!} + \ldots \frac{\beta_{m-2}}{2!} + 2\beta_m &= \frac{1}{m!}.
\end{aligned}$$

What we want to prove in Claim 1 can be written as

$$\frac{\beta_1}{m!} + \frac{\beta_3}{(m-2)!} + \ldots + \frac{\beta_{m-2}}{3!} + \beta_m(1 + \frac{1}{n-m}) = \frac{1}{(m+1)!}$$

vi) Reduction 6: A universal system of linear equations

Let $\theta_1, \theta_3, \theta_5, \ldots$ be the unique sequence satisfying the following system of infinitely many linear equations

$$\begin{aligned} 2\theta_1 &= 1,\\ \frac{\theta_1}{2!} + 2\theta_3 &= 0,\\ \frac{\theta_1}{4!} + \frac{\theta_3}{2!} + 2\theta_5 &= 0,\\ &\ldots, \end{aligned}$$

Then, for any sequence $c_1, c_3, c_5, \ldots$, the unique solution to

$$\begin{aligned} 2z_1 &= c_1,\\ \frac{z_1}{2!} + 2z_3 &= c_3,\\ \frac{z_1}{4!} + \frac{z_3}{2!} + 2z_5 &= c_5,\\ &\ldots, \end{aligned}$$

is

$$\begin{aligned} z_1 &= c_1\theta_1,\\ z_3 &= c_3\theta_1 + c_1\theta_3,\\ z_5 &= c_5\theta_1 + c_3\theta_3 + c_5\theta_1,\\ \ldots \end{aligned}$$

vii) Reduction 7: Let $\alpha_1, \alpha_3, \ldots$ be the unique sequence satisfying the following system

$$\begin{aligned} 2\alpha_1 &= \frac{1}{1!},\\ \frac{\alpha_1}{2!} + 2\alpha_3 &= \frac{1}{3!},\\ \frac{\alpha_1}{4!} + \frac{\alpha_3}{2!} + 2\alpha_5 &= \frac{1}{5!},\\ &\ldots \end{aligned}$$

Then it is easy to see that for β_j in Reduction 4:

$$\beta_j = \alpha_j - \frac{1}{n}\theta_j,$$

for all $j = 1, 3, \ldots, m$, and what we wanted to prove in Claim 1 becomes

$$-\frac{1}{n}\left(\frac{\theta_1}{m!} + \frac{\theta_3}{(m-2)!} + \ldots + \frac{\theta_{m-2}}{3!} + \frac{\theta_m}{1!} - \frac{\theta_m}{m}\right) + \frac{1}{n-m}\left(\alpha_m - \frac{\theta_m}{m}\right) = 0.$$

Hence Claim 1 is proved if we can prove the following Claim

Claim 2: For any $m \in \mathbb{N}$, m odd then the following conclusions are true

$$\frac{\theta_1}{m!} + \frac{\theta_3}{(m-2)!} + \ldots + \frac{\theta_{m-2}}{3!} + \frac{\theta_m}{1!} - \frac{\theta_m}{m} = 0, \tag{5.5}$$

and

$$\alpha_m - \frac{\theta_m}{m} = 0. \tag{5.6}$$

viii) Proof of Claim 2:

Define a formal series

$$\theta(t) = \theta_1 - t^2\theta_3 + t^4\theta_5 - t^6\theta_7 + \dots$$

From the Reduction 6:

$$1 = \theta(t).(2 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} \dots) = \theta(t).(1 + \cos t).$$

Hence

$$\theta(t) = \frac{1}{1 + \cos t}.$$

Similarly, if we define

$$\alpha(t) = t\alpha_1 - t^3\alpha_3 + t^5\alpha_5 \dots$$

then from Reduction 7

$$\alpha(t) = \frac{\sin t}{1 + \cos t}.$$

It follows that

$$\frac{d\alpha}{dt} = \theta(t),$$

which proves (5.6).

From Reductions 6 and 7 we have

$$\alpha_m = \frac{\theta_1}{m!} + \frac{\theta_3}{(m-2)!} + \dots + \frac{\theta_{m-2}}{3!} + \frac{\theta_m}{1!}.$$

This equality and (5.6) imply (5.5). Hence we completed the proof of Lemma 3. □

Lemma 4. *Let $n \geq 3$ be an odd integer. Let $a_0, \dots, a_n$ be a solution of the system of linear equations*

$$L_j(a_0, a_1, \dots, a_n) = 0$$

for all $j = 0, 1, 2, \dots, n-1$. Then

$$\sum_{j=2}^{n} (-1)^j a_j = 0.$$

Proof. To prove the equality we need only to take the difference between the sum of odd-th equations and the sum of even-th equations. □

6. (1,1)-regularization for exceptional cases: $n =$odd

In this Section we show how to (1,1)-regularize the maps k_F in exceptional cases when $n \geq 3$ is odd . Recall that $F(z) = a_n z^n + a_{n-1}z^{n-1} + \dots + a_1 z + a_0$ is a polynomial ($a_n \neq 0$).

Let Y be the manifold constructed in Section 4 in [2]. If $a_0 = \frac{2}{m+1}$ for some $m = 0, 1, 2, \dots$, let $\widehat{Y}$ denote the manifold constructed by blowing up the manifold Y at the points $r_0, q_0, r_1, q_1, \dots, r_m, q_m$ as in the proof of Theorem 1. Otherwise,

i.e. if $a_0\neq \frac{2}{m+1}$ for any $m=0,1,2,\ldots$, let $\widehat{Y}$ denote the space Y itself. Let us denote $Y_1=\widehat{Y}$. Let us also denote some more notations

$$\begin{aligned} ep_0 &= \frac{1}{a_n}\in P_{n-1},\ ip_0=\frac{1}{a_n}\in P_{n-1},\\ ep_1 &= -\frac{a_{n-1}}{a_n^2}\in P_n,\ ip_1=\frac{-(n-1)a_n+a_{n-1}}{a_n^2}\in P_n. \end{aligned}$$

The above equations mean that ep_0 is a point of P_{n-1} with local coordinate $\frac{1}{a_n}$, and so on. Here "ep" means "exceptional point" that is points which is the image of some exceptional curves, and "ip" means "indeterminate point" that is points which blowups to some curves.

For convenience we recall the action of the induced map $k_{Y_1}:Y_1\rightarrow Y_1$ (see [2]):

$$\begin{aligned} k_{Y_1}:\ C_1,C_2,P_1,P_2,\ldots,P_{n-3} &\mapsto ep_1\in P_n,\\ k_{Y_1}^{-1}:\ C_1,P_1,P_2,\ldots,P_{n-3} &\mapsto ip_1\in P_n, \end{aligned}$$

and $k_{Y_1}:P_n\longleftrightarrow P_{n-2}$ with

$$\begin{aligned} P_n\ni u &\mapsto \frac{1}{-a_n^2u-(n-1)a_n+a_{n-1}}\in P_{n-2},\\ P_{n-2}\ni u &\mapsto -\frac{1}{a_n^2u}-\frac{a_{n-1}}{a_n}\in P_n. \end{aligned}$$

We will prove the following result

Theorem 4. *Let $n=deg(F)\geq 3$ be odd. Let Y be the manifold constructed in Section 4 in [2]. Let $L_j:\mathbb{C}^{n+1}\rightarrow\mathbb{C}$ ($j=0,1,\ldots,n$) be linear functions defined in Section 3. Then there exists $0\leq j\leq n-2$ such that*

$$L_i(a_0,a_1,\ldots,a_n)=0$$

for all $0\leq i\leq j$. Moreover, we choose j as large as possible, that is either $j=n-2$ or $L_{j+1}(a_0,a_1,\ldots,a_n)\neq 0$.

If $a_0=\frac{2}{m+1}$ for some $m=0,1,2,\ldots$, let $\widehat{Y}$ denote the manifold constructed by blowing up the manifold Y at the points $r_0,q_0,r_1,q_1,\ldots,r_m,q_m$ as in the proof of Theorem 1. Otherwise, i.e. if $a_0\neq\frac{2}{m+1}$ for any $m=0,1,2,\ldots$, let $\widehat{Y}$ denote the space Y itself.

We have one of the following alternatives

Case 1: $j<n-2$. Then there exists spaces $Y_1=\widehat{Y},Y_2,\ldots,Y_j,Y_{j+1}=Z$, where $Y_{i+1}\rightarrow Y_i$ is a one point-blowup for $i\leq j-1$, such that the induced map $k_Z:Z\rightarrow Z$ is A.S.

Case 2: $j=n-2$.

Subcase 2.1: $a_0\neq\frac{n+1}{2}+\frac{m}{2(1+m)}$ for any $m=0,1,2,\ldots$. Then there exists spaces $Y_1=\widehat{Y},Y_2,\ldots,Y_n=Z$, where $Y_{i+1}\rightarrow Y_i$ is a one point-blowup for $i\leq n-1$, such that the induced map $k_Z:Z\rightarrow Z$ is A.S.

Case 2.2: $a_0=\frac{n+1}{2}+\frac{m}{2(1+m)}$ for some $m=0,1,2,\ldots$. Let Y_n be the space in Subcase 2.1. Then there exists spaces $Y_n,Y_{n+1},\ldots,Y_{n+2m+2}=Z$, where $Y_{n+i+1}\rightarrow Y_{n+i}$ is a one point-blowup for $i\leq 2m+1$, such that the induced map $k_Z:Z\rightarrow Z$ is A.S.

The proof of Theorem 4 is divided into some steps.

Lemma 5. *Let $1 \le m \le n-3$ be an integer. Assume that*

$$L_j(a_0, a_1, \ldots, a_n) = 0 \tag{6.1}$$

for all $j = 0, 1, 2, \ldots, m$, and

$$L_{m+1}(a_0, a_1, \ldots, a_n) \neq 0.$$

Construct a sequence of spaces Y_j ($2 \le j \le m+1$) by induction as follows: Y_j is the blowup of Y_{j-1} at a point $ip_{j-1} = ep_{j-1} \in P_{n-1+j-1}$, and P_{n-1+j} is the exceptional fiber of the blowup $Y_j \to Y_{j-1}$. Here the points ip_j and ep_j ($1 \le j \le m+1$) are defined as

$$ip_j = \frac{c_j + d_j}{a_n^2}, \; ep_j = \frac{\gamma_j + c_j}{a_n^2},$$

where $-a_n^2 u + c_j$ is the coefficient of s^j of the Taylor expansion of the function

$$\frac{1+s}{ip_0 + ip_1 s + \ldots + ip_{j-1}s^{j-1} + s^j u}$$

near $s = 0$, d_j is the coefficient of s^j in the polynomial $s^n F(-1-\frac{1}{s})$, γ_j is the coefficient of s^j in the polynomial $-(1+s)s^n F(\frac{1}{s})$. Moreover for small values of s:

$$\begin{aligned}
\frac{1+s}{ip_0 + ip_1 s + \ldots + ip_{j-1}s^{j-1} + s^j u} + s^n F(-1-\frac{1}{s}) &= (-a_n^2 u + c_j)s^j + d_j s^j + O(s^{j+1}), \\
\frac{1+s}{ip_0 + ip_1 s + \ldots + ip_{j-1}s^{j-1} + s^j u} - (1+s)s^n F(\frac{1}{s}) &= (-a_n^2 u + c_j)s^j + \gamma_j s^j + O(s^{j+1}).
\end{aligned}$$

Then the induced map $k_{Y_{m+1}} : Y_{m+1} \to Y_{m+1}$ is A.S., and acts as follows:

$$\begin{aligned}
k_{Y_{m+1}} : \; C_1, C_2, P_1, \ldots, P_{n-1-(m+1)-1} &\mapsto ep_{m+1} \in P_{n-1+m+1}, \\
k^{-1}_{Y_{m+1}} : C_1, P_1, \ldots, P_{n-1-(m+1)-1} &\mapsto ip_{m+1} \in P_{n-1+m+1},
\end{aligned}$$

and $k_{Y_{m+1}} : P_{n-1+m+1} \longleftrightarrow P_{n-1-(m+1)}$ is

$$\begin{aligned}
k_{Y_{m+1}} : \; P_{n-1+m+1} \ni u &\mapsto \frac{(-1)^{n-(m+1)}}{-a_n^2 u + d_{m+1} + c_{m+1}} \in P_{n-1-(m+1)}, \\
k_{Y_{m+1}} : \; P_{n-1-(m+1)} \ni u &\mapsto \frac{(-1)^{n-(m+1)}}{-a_n^2 u} + \frac{c_{m+1} + \gamma_{m+1}}{a_n^2} \in P_{n-1+(m+1)}.
\end{aligned}$$

In these formulas we choose the coordinate projection at P_{n-1+j} as

$$(s,u) \mapsto [s^n(ip_0 + ip_1 s + \ldots + ip_{j-1}s^{j-1} + s^j u) : 1 : s^{n-1}(ip_0 + ip_1 s + \ldots + ip_{j-1}s^{j-1} + s^j u)].$$

Proof. 1) Step 1: We will prove by induction on m that if (6.2) is satisfied for $0 \le j \le m$ then a sequence of spaces Y_j ($1 \le j \le m+1$) exists and satisfies all the conclusion of Lemma 5 except the conclusion that $k_{Y_{m+1}}$ is A.S.

Proof: Assume by induction that spaces $Y_0, Y_1, \ldots, Y_l$ ($l < m+1$) was constructed with the following properties: Y_j is the blowup of Y_{j-1} at a point $ip_{j-1} = ep_{j-1} \in P_{n-1+j-1}$, and P_{n-1+j} is the exceptional fiber of the blowup $Y_j \to Y_{j-1}$. Here the points ip_j and ep_j ($1 \le j \le l$) are defined as

$$ip_j = \frac{c_j + d_j}{a_n^2}, \; ep_j = \frac{\gamma_j + c_j}{a_n^2},$$

where $-a_n^2u+c_j$ is the coefficient of s^j of the Taylor expansion of the function

$$\frac{1+s}{ip_0+ip_1s+\ldots+ip_{j-1}s^{j-1}+s^ju}$$

near $s=0$, d_j is the coefficient of s^j in the polynomial $s^nF(-1-\frac{1}{s})$, γ_j is the coefficient of s^j in the polynomial $-(1+s)s^nF(\frac{1}{s})$. Moreover for small values of s:

$$\begin{aligned}\frac{1+s}{ip_0+ip_1s+\ldots+ip_{j-1}s^{j-1}+s^ju}+s^nF(-1-\frac{1}{s}) &= (-a_n^2u+c_j)s^j+d_js^j+O(s^{j+1}),\\ \frac{1+s}{ip_0+ip_1s+\ldots+ip_{j-1}s^{j-1}+s^ju}-(1+s)s^nF(\frac{1}{s}) &= (-a_n^2u+c_j)s^j+\gamma_js^j+O(s^{j+1}).\end{aligned}$$

The induced map $k_{Y_l}:Y_l\rightarrow Y_l$ acts as follows:

$$\begin{aligned}k_{Y_l}:\ C_1,C_2,P_1,\ldots,P_{n-1-l-1} &\mapsto ep_l\in P_{n-1+l},\\ k_{Y_l}^{-1}:C_1,P_1,\ldots,P_{n-1-l-1} &\mapsto ip_l\in P_{n-1+l},\end{aligned}$$

and $k_{Y_l}:P_{n-1+l}\longleftrightarrow P_{n-1-l}$ is

$$\begin{aligned}k_{Y_l}:\ P_{n-1+l}\ni u &\mapsto \frac{(-1)^{n-l}}{-a_n^2u+d_l+c_l}\in P_{n-1-l},\\ k_{Y_l}:\ P_{n-1-l}\ni u &\mapsto \frac{(-1)^{n-l}}{-a_n^2u}+\frac{c_l+\gamma_l}{a_n^2}\in P_{n-1+l}.\end{aligned}$$

The starting point $l=0$ can be easily checked to satisfy the above conditions.

i) Claim 1: The following two facts are equivalent

Fact 1: $ep_j=ip_j$ for all $0\leq j\leq l$.

Fact 2: $L_j(a_0,a_1,\ldots,a_n)=0$ for all $0\leq j\leq l$.

Proof of Claim 1: From the definition of ep_j and ip_j it is not much difficult to check Claim 1 (the reader may check with concrete examples to see how this works).

ii) Claim 2: k_{Y_l} is not A.S.

Proof of Claim 2: From Claim 1 and the action of k_{Y_l} we see that

$$k_{Y_l}:\ C_1,C_2,P_1,\ldots,P_{n-1-l-1}\mapsto ep_l=ip_l\in P_{n-1+l},$$

which is an indeterminate point of k_{Y_l}. Hence k_{Y_l} is not A.S.

iii) Claim 3: Let Y_{l+1} be the blowup of Y_l at the point $ep_l=ip_l\in P_{n-1+l}$, and let $P_{n-1+l+1}$ be the exceptional fiber of this blowup. Choose the coordinate projection at $P_{n-1+l+1}$ as described in the statement of Lemma 5. Then the action of the induced map $k_{Y_{l+1}}:Y_{l+1}\rightarrow Y_{l+1}$ is

$$\begin{aligned}k_{Y_{l+1}}:\ C_1,C_2,P_1,\ldots,P_{n-1-(l+1)-1} &\mapsto ep_{l+1}\in P_{n-1+l+1},\\ k_{Y_{l+1}}^{-1}:C_1,P_1,\ldots,P_{n-1-(l+1)-1} &\mapsto ip_{l+1}\in P_{n-1+l+1},\end{aligned}$$

and $k_{Y_{l+1}}:P_{n-1+l+1}\longleftrightarrow P_{n-1-(l+1)}$ is

$$\begin{aligned}k_{Y_{l+1}}:\ P_{n-1+l+1}\ni u &\mapsto \frac{(-1)^{n-(l+1)}}{-a_n^2u+d_{l+1}+c_{l+1}}\in P_{n-1-(l+1)},\\ k_{Y_{l+1}}:\ P_{n-1-(l+1)}\ni u &\mapsto \frac{(-1)^{n-(l+1)}}{-a_n^2u}+\frac{c_{l+1}+\gamma_{l+1}}{a_n^2}\in P_{n-1+(l+1)}.\end{aligned}$$

Proof of Claim 3: First we compute the image of a generic point $u\in P_{n-1+l+1}$ under the map $k_{Y_{l+1}}$. Use the formula

$$k_{Y_{l+1}}[u]=\lim_{s\rightarrow 0}k[s^n(ip_0+ip_1s+\ldots+ip_ls^l+s^{l+1}u):1:s^{n-1}(ip_0+ip_1s+\ldots+ip_ls^l+s^{l+1}u)],$$

and the fact that

$$\begin{aligned}\frac{1+s}{ip_0+ip_1s+\ldots+ip_{j-1}s^{j-1}+s^ju}+s^nF(-1-\frac{1}{s}) &= (-a_n^2u+c_j)s^j+d_js^j+O(s^{j+1}),\\ \frac{1+s}{ip_0+ip_1s+\ldots+ip_{j-1}s^{j-1}+s^ju}-(1+s)s^nF(\frac{1}{s}) &= (-a_n^2u+c_j)s^j+\gamma_js^j+O(s^{j+1}).\end{aligned}$$

for all $j\le l+1$, it is not hard to see that

$$k_{Y_{l+1}}:\ P_{n-1+l+1}\ni u \quad\mapsto\quad \frac{(-1)^{n-(l+1)}}{-a_n^2u+d_{l+1}+c_{l+1}}\in P_{n-1-(l+1)}.$$

Now we compute the image of a generic point $u\in P_{n-1+(l+1)}$ under the map $k_{Y_{l+1}}^{-1}$. Use the formula

$$k_{Y_{l+1}}^{-1}[u]=\lim_{s\to 0}k^{-1}[s^n(ip_0+ip_1s+\ldots+ip_ls^l+s^{l+1}u):1:s^{n-1}(ip_0+ip_1s+\ldots+ip_ls^l+s^{l+1}u)],$$

and arguing as above, it is not hard to see that

$$k_{Y_{l+1}}^{-1}:\ P_{n-1+l+1}\ni u \quad\mapsto\quad \frac{(-1)^{n-(l+1)}}{-a_n^2u+\gamma_{l+1}+c_{l+1}}\in P_{n-1-(l+1)}.$$

Then the image of a generic point $u\in P_{n-1-(l+1)}$ using $k_{Y_{l+1}}=(k_{Y_{l+1}}^{-1})^{-1}$ can be computed as follows

$$k_{Y_{l+1}}:\ P_{n-1-(l+1)}\ni u\mapsto \frac{(-1)^{n-(l+1)}}{-a_n^2u}+\frac{\gamma_{l+1}+c_{l+1}}{a_n^2}\in P_{n-1+l+1}.$$

(The reason why we computed $k_{Y_{l+1}}:P_{n-1-(l+1)}\to P_{n-1+(l+1)}$ via $k_{Y_{l+1}}^{-1}:P_{n-1+(l+1)}\to P_{n-1-(l+1)}$ is because the formula for coordinate projection of $P_{n-1-(l+1)}$ is much more simpler than that of $P_{n-1+(l+1)}$.) Hence $k_{Y_{l+1}}:P_{n-1+l+1}\longleftrightarrow P_{n-1-(l+1)}$. This fact, and the inductional assumption that

$$k_{Y_l}:\ C_1,C_2,P_1,\ldots,P_{n-1-l-1}\quad\mapsto\quad ep_l\in P_{n-1+l},$$

imply that $C_1,C_2,P_1,\ldots,P_{n-1-(l+1)-1}$ are exceptional curves for $k_{Y_{l+1}}$, and hence their images must be the points lie in $P_{n-1+(l+1)}$ which is indeterminate points for $k_{Y_{l+1}}^{-1}$. From the formula for $k_{Y_{l+1}}^{-1}:P_{n-1+(l+1)}\to P_{n-1-(l+1)}$ which we found above, there is only such a point which is exactly ep_{l+1}. Hence

$$k_{Y_{l+1}}:\ C_1,C_2,P_1,\ldots,P_{n-1-(l+1)-1}\quad\mapsto\quad ep_{l+1}\in P_{n-1+l+1}.$$

Similarly

$$k_{Y_{l+1}}^{-1}:C_1,P_1,\ldots,P_{n-1-(l+1)-1}\quad\mapsto\quad ip_{l+1}\in P_{n-1+l+1}.$$

Using the above Claims we complete the proof of Step 1.

2) Step 2: Completion of the proof of Lemma 5: In Step 1 we showed that a sequence of spaces Y_j $(1\le j\le m+1)$ exists and satisfies all the conclusion of Lemma 5 except the conclusion that $k_{Y_{m+1}}$ is A.S. Now we show that $k_{Y_{m+1}}$ is A.S.

From Step 1 we have

$$k_{Y_{m+1}}^2:P_{n-1+m+1}\ni u\mapsto u+ep_{m+1}-ip_{m+1}\in P_{n-1+m+1}.$$

Hence the orbit of the exceptional curves are

$$k_{Y_{m+1}}^{2l+1}:\ C_1,C_2,P_1,\ldots,P_{n-1-(m+1)-1}\mapsto ep_{m+1}+l(ep_{m+1}-ip_{m+1})\in P_{n-1+m+1}$$

never land on the indeterminate point $ip_{m+1} \in P_{n-1+m+1}$, since $ep_{m+1} \neq ip_{m+1}$ as can be easily seen from Claim 1 in Step 1 and our assumption that $L_{m+1}(a_0, a_1, \ldots, a_n) \neq 0$. This implies that $k_{Y_{m+1}}$ is A.S. $\square$

Lemma 6. *Assume that*

$$L_j(a_0, a_1, \ldots, a_n) = 0 \tag{6.2}$$

for all $j = 0, 1, 2, \ldots, n-3$. Construct the spaces $Y_1, \ldots, Y_{n-2}$ as described in Lemma 5.

Case 1: $L_{n-2}(a_0, a_1, \ldots, a_n) \neq 0$. Then the induced map $k_{Y_{n-2}}$ is A.S.

Case 2: $L_{n-2}(a_0, a_1, \ldots, a_n) = 0$. Then the induced map $k_{Y_{n-2}}$ is not A.S, and $ep_{n-2} = ip_{n-2}$. Construct the space Y_{n-1} as the blowup of Y_{n-1} at the point ep_{n-2}, and call $P_{n-1+n-1}$ the exceptional fiber of this blowup Y_{n-1}. Then the action of the induced map $k_{Y_{n-1}}$ is

$$k_{Y_{n-1}}: \ P_{n-1+n-1} \quad \longleftrightarrow \quad C_1, \ C_2 \mapsto ep_{n-1} \mapsto [0:0:1] = e_2,$$

where $ep_{n-1} \in P_{n-1+n-1}$ is constructed in the same way as $ep_1, \ldots, ep_{n-2}$. The map $k_{Y_{n-1}}$ has no indeterminate point lying in $P_{n-1+n-1}$, but it is not A.S.

Let Y_n be the blowup of Y_{n-1} at two points $ep_{n-1} \in P_{n-1+n-1}$ and $e_2 = [0:0:1]$, call P_{n-1+n} and E_2 the exceptional fibers of this blowup $Y_n \to Y_{n-1}$. Let the coordinate projection at P_{n-1+n} as

$$(s, u) \mapsto [s^n(ep_0 + ep_1 s + \ldots + ep_{n-1}s^{n-1} + s^n u) : 1 : s^{n-1}(ep_0 + ep_1 s + \ldots + ep_{n-1}s^{n-1} + s^n u)],$$

and the coordinate projection at E_2 is

$$(s, u) \mapsto [s : su : 1].$$

(Recall that we do not have a point ip_{n-1}, however we do have the points $ip_0 = ep_0, ip_1 = ep_1, \ldots, ip_{n-2} = ep_{n-2}$.) Under these coordinates then the induced map $k_{Y_n}: Y_n \to Y_n$ is

$$k_{Y_n}: C_2 \mapsto ep_n \in P_{n-1+n},$$

and $k_{Y_n}: P_{n-1+n} \longleftrightarrow E_2$ as

$$\begin{aligned} k_{Y_n}: \ P_{n-1+n} \ni u \quad &\mapsto \quad -a_n^2 u + c_n + d_n - (n+1) \in E_2, \\ k_{Y_n}: \ E_2 \ni u \quad &\mapsto \quad \frac{-u + c_n + \gamma_n + 1}{a_n^2} \in P_{n-1+n}. \end{aligned}$$

Here the constants ep_n, c_n, d_n, γ_n are

$$ep_n = \frac{c_n + \gamma_n}{a_n^2},$$

and d_n is the coefficient of s^n in the polynomial $s^n F(-1 - \frac{1}{s})$, γ_n is the coefficient of s^n in the polynomial $(1+s)s^n F(\frac{1}{s})$, and when using Taylor's expansion for s small enough

$$\begin{aligned} \frac{1+s}{ep_0 + ep_1 s + \ldots + ep_{n-1}s^{n-1} + s^n u} + s^n F(-1 - \frac{1}{s}) \quad &= \quad (-a_n^2 u + c_n + d_n)s^n + O(s^{n+1}). \\ \frac{1+s}{ep_0 + ep_1 s + \ldots + ep_{n-1}s^{n-1} + s^n u} - (1+s)s^n F(-1 - \frac{1}{s}) \quad &= \quad (-a_n^2 u + c_n + \gamma_n)s^n + O(s^{n+1}). \end{aligned}$$

Moreover $0 \in E_2$ is the only indeterminate point lying in E_2 of k_{Y_n}.

Subcase 2.1: $a_0 \neq \frac{n+1}{2} + \frac{l}{2(1+l)}$ for any $l = 0, 1, 2, \ldots$. Then the induced map $k_{Y_n}: Y_n \to Y_n$ is A.S.

Subcase 2.2: $a_0 = \frac{n+1}{2} + \frac{l}{2(1+l)}$ *for some* $l = 0, 1, 2, \ldots$. *Let* Z *be the space constructed by blowing up* Y_n *at points*

$$\begin{aligned} ep_n &\in P_{n-1+n},\ k_{Y_n}(ep_n) \in E_2, \\ k^2_{Y_n}(ep_n) &\in P_{n-1+n}, \\ &\ldots \\ k^{2l}_{Y_n}(ep_n) &\in P_{n-1+n},\ k^{2l+1}_{Y_n}(ep_n) = 0 \in E_2. \end{aligned}$$

Then the induced map $k_Z : Z \to Z$ *is A.S.*

Proof. Case 1 and the action of the induced map $k_{Y_{n-1}} \to k_{Y_{n-1}}$ can be proved as in Lemma 5.

Now the action

$$k_{Y_n} :\ P_{n-1+n} \ni u \mapsto -a_n^2 u + c_n + d_n - (n+1) \in E_2 \tag{6.3}$$

can be computed as in Lemma 5. In the same way we can compute

$$k^{-1}_{Y_n} :\ P_{n-1+n} \ni u \mapsto -a_n^2 u + c_n + \gamma_n - 1 \in E_2,$$

hence using $k_{Y_n} = (k^{-1}_{Y_n})^{-1}$ we get

$$k_{Y_n} :\ E_2 \ni u \mapsto \frac{-u + c_n + \gamma_n + 1}{a_n^2} \in P_{n-1+n}. \tag{6.4}$$

That $0 \in E_2$ is the only indeterminate point lying in E_2 of k_{Y_n} is not hard to see, and that the image of C_2 is a point $ep_n \in P_{n-1+n}$ can be proved by the same argument in the proof of Lemma 5. Now we compute ep_n. We have $C_2 \cap E_2 = 1 \in E_2$ which is not an indeterminate point of k_{Y_n}, hence using (6.4)

$$ep_n = k_{Y_n}(C_2) = k_{Y_n}([1]_{E_2}) = \frac{c_n + \gamma_n}{a_n^2}.$$

From (6.3) and (6.4) we get

$$k^2_{Y_n} :\ E_2 \ni u \mapsto u - \gamma_n + d_n - (n+2).$$

Then

$$k_{Y_n}(ep_n) = -\gamma_n + d_n - (n+1) \in E_2.$$

Using the formulas of γ_n and d_n, and using Lemma 4 we have $d_n - \gamma_n = 2a_0$. Hence the orbit of C_2 is

$$k^{2l+2}_{Y_n} : C_2 \mapsto 2a_0(l+1) - (n+1)(l+1) - l \in E_2.$$

This orbit lands on the indeterminate point $0 \in E_2$ iff

$$2a_0(l+1) - (n+1)(l+1) - l = 0$$

for some $l = 0, 1, 2, \ldots$. Then Case 2 easy follows. □

7. Proof of Theorem 2

In this section we prove Theorem 2. Let Z be the spaces constructed in Theorem 4. Since the map $k_Z : Z \to Z$ is A.S., we obtain $\delta(k_F)$ as the spectral radius of k_Z^*.

1. Case $n = deg(F)$ is odd; $a_0 \neq 2/(m+1)$ for any $m = 0, 1, 2, \ldots$; $L_i(a_0, a_1, \ldots, a_n) = 0$ for any $0 \leq i \leq h$, $L_{h+1}(a_0, \ldots, a_n) \neq 0$ where $0 \leq h < n-2$. As noted before, in this case h must be an even integer.

Lemma 7. *The spectral radius of $k_Z^*: Pic(Z) \to Pic(Z)$ is the largest real root of the polynomial $x^3 - nx^2 - (n+1-h)x - 1$.*

Proof. Let Z be the space constructed in Theorem 4. Let $H_Z, E_1, Q, P_1, \ldots, P_{n-1+h+1}$ be a basis for $Pic(Z)$ (see convenience in the proof of Theorem 1). In this basis then

$$\begin{aligned}
C_1 &= H_Z - E_1 - Q - \sum_{j=1}^{n-1}(j+1)P_j - n\sum_{j=1}^{h+1} P_{n-1+j},\\
C_2 &= H_Z,\ C_3 = H_Z - E_1 - Q - \sum_{j=1}^{n-1} jP_j - (n-1)\sum_{j=1}^{h+1} P_{n-1+j},\\
C_4 &= 2H_Z - E_1 - 2Q - \sum_{j=1}^{n-1} jP_j - (n-1)\sum_{j=1}^{h+1} P_{n-1+j}.
\end{aligned}$$

As in the proof of Theorem 1, $k_Z^*: Pic(Z) \to Pic(Z)$ acts as

$$\begin{aligned}
k_Z^*(H_Z) &= (2n+1)H_Z - nE_1 - (n+1)Q - (n+1)\sum_{j=1}^{n-1} jP_j - \sum_{j=1}^{h+1}(n^2-1+j)P_{n-1+j},\\
k_Z^*(E_1) &= E_1,\ k_Z^*(Q) = H_Z - E_1 - Q - \sum_{j=1}^{n-1} jP_j - (n-1)\sum_{j=1}^{h+1} P_{n-1+j},\\
k_Z^*(P_{n-1-j}) &= 0,\ j > h+1,\ k_Z^*(P_{n-1-(h+1)}) = P_{n-1+h+1},\\
k_Z^*(P_{n-1-j}) &= P_{n-1+j},\ j = 0, \ldots, h,\ k_Z^*(P_{n-1+j}) = P_{n-1-j},\ j = 0, \ldots, h,\\
k_Z^*(P_{n-1+h+1}) &= C_1 + C_2 + P_1 + \ldots + P_{n-1-(h+1)}\\
&= 2H_Z - \sum_{j=1}^{n-1-(h+1)} jP_j - \sum_{j=n-1-h}^{n-1}(j+1)P_j - n\sum_{j=1}^{h+1} P_{n-1+j}.
\end{aligned}$$

The spectral radius of k_Z^* can be computed as the greatest real zero of the characteristic polynomial of the matrix representation of k_Z^* restricted to $\{H_Z, Q, P_{n-1-h-1}, P_{n-1+h+1}\}$ which is

$$M_1 = \begin{pmatrix}
2n+1 & -(n+1) & -(n+1)(n-1-h-1) & -(n^2-1+h+1)\\
1 & -1 & -(n-1-h-1) & -(n-1)\\
0 & 0 & 0 & 1\\
2 & -1 & -(n-1-h-1) & -n
\end{pmatrix}$$

The characteristic polynomial $P(x) = det(M_1 - xI)$ of M_1 is

$$P(x) = -x(x^3 - nx^2 - (n+1-h)x - 1).$$

From this the conclusions of Lemma 7 follow. □

2. Case $n = deg(F)$ is odd; $a_0 = 2/(m+1)$ for some $m = 0, 1, 2, \ldots$; $L_i(a_0, a_1, \ldots, a_n) = 0$ for any $0 \le i \le h$, $L_{h+1}(a_0, \ldots, a_n) \neq 0$ where $0 \le h < n-2$. As noted before, in this case h must be an even integer.

Lemma 8. *The spectral radius of $k_Z^*: Pic(Z) \to Pic(Z)$ is the largest real root of the polynomial $x^{2m+1}(x^3 - nx^2 - (n-h+1)x - 1) + x^3 + x^2 + nx + n - h - 1$.*

Proof. Let Z be the space constructed in Theorem 4. Let $H_Z, E_1, Q, P_1, \ldots, P_{n-1+h+1}$, $Q_1, \ldots, Q_m, R_0, R_1, \ldots, R_m$ be a basis for $Pic(Z)$. In this basis then

$$\begin{aligned}
C_1 &= H_Z - E_1 - Q - \sum_{j=1}^{n-1}(j+1)P_j - n\sum_{j=1}^{h+1} P_{n-1+j} - \sum_{j=1}^{m} Q_j,\\
C_2 &= H_Z - R_m,\\
C_3 &= H_Z - E_1 - Q - \sum_{j=1}^{n-1} jP_j - (n-1)\sum_{j=1}^{h+1} P_{n-1+j} - \sum_{j=1}^{m} Q_j - \sum_{j=0}^{m} R_j,\\
C_4 &= 2H_Z - E_1 - 2Q - \sum_{j=1}^{n-1} jP_j - (n-1)\sum_{j=1}^{h+1} P_{n-1+j} - 2\sum_{j=1}^{m} Q_j - R_m.
\end{aligned}$$

Then $k_Z^* : Pic(Z) \to Pic(Z)$ acts as

$$\begin{aligned}
k_Z^*(H_Z) &= (2n+1)H_Z - nE_1 - (n+1)Q - (n+1)\sum_{j=1}^{n-1} jP_j - \sum_{j=1}^{h+1}(n^2-1+j)P_{n-1+j}\\
&\quad -(n+1)\sum_{j=1}^{m} Q_j - (n+1)R_m,\\
k_Z^*(E_1) &= E_1,\\
k_Z^*(Q) &= H_Z - E_1 - Q - \sum_{j=1}^{n-1} jP_j - (n-1)\sum_{j=1}^{h+1} P_{n-1+j} - \sum_{j=1}^{m} Q_j - \sum_{j=0}^{m} R_j,\\
k_Z^*(P_{n-1-j}) &= 0,\ j > h+1,\ k_Z^*(P_{n-1-(h+1)}) = P_{n-1+h+1},\\
k_Z^*(P_{n-1-j}) &= P_{n-1+j},\ j = 0, \ldots, h,\ k_Z^*(P_{n-1+j}) = P_{n-1-j},\ j = 0, \ldots, h,\\
k_Z^*(P_{n-1+h+1}) &= 2H_Z - \sum_{j=1}^{n-1-(h+1)} jP_j - \sum_{j=n-1-h}^{n-1} (j+1)P_j - n\sum_{j=1}^{h+1} P_{n-1+j} - \sum_{j=1}^{m} Q_j - R_m,\\
k_Z^*(R_0) &= 2H_Z - E_1 - 2Q - \sum_{j=1}^{n-1} jP_j - (n-1)\sum_{j=1}^{h+1} P_{n-1+j} - 2\sum_{j=1}^{m} Q_j - R_m,\\
k_Z^*(R_j) &= Q_j,\ 1 \le j \le m,\ k_Z^*(Q_j) = R_{j-1},\ 1 \le j \le m.
\end{aligned}$$

The spectral radius of k_Z^* can be computed as the greatest real zero of the characteristic polynomial of the matrix representation M_1 of k_Z^* restricted to $\{H_Z, Q, P_{n-1-h-1}, P_{n-1+h+1}$, $Q_1, \ldots, Q_m,\ R_0, R_1, \ldots, R_m\}$ which is

$$P(x) = -x[x^{2m+1}(x^3 - nx^2 - (n-h+1)x - 1) + x^3 + x^2 + nx + n - h - 1].$$

From this the conclusions of Lemma 8 follow. □

3. Case $n = deg(F)$ is odd; $a_0 \neq 2/(m+1)$ for any $m = 0, 1, 2, \ldots$; $a_0 \neq \frac{n+1}{2} + \frac{l}{2(l+1)}$ for any $l = 0, 1, 2, \ldots$; $L_i(a_0, a_1, \ldots, a_n) = 0$ for any $0 \le i \le n-2$.

Lemma 9. *The spectral radius of $k_Z^* : Pic(Z) \to Pic(Z)$ is the largest real root of the polynomial $x^3 - nx^2 - 2x - 1$.*

Proof. Let Z be the space constructed in Theorem 4. Let $H_Z, E_1, Q, P_1, \ldots, P_{n-1+n}, E_2$ be a basis for $Pic(Z)$. Then $k_Z^* : Pic(Z) \rightarrow Pic(Z)$ acts as

$$\begin{aligned}
k_Z^*(H_Z) &= (2n+1)H_Z - nE_1 - (n+1)Q - (n+1)\sum_{j=1}^{n-1} jP_j \\
&\quad - \sum_{j=1}^{n-1}(n^2-1+j)P_{n-1+j} - (n^2+n-2)P_{n-1+n} - nE_2, \\
k_Z^*(E_1) &= E_1,\ k_Z^*(Q) = H_Z - E_1 - Q - \sum_{j=1}^{n-1} jP_j - (n-1)\sum_{j=1}^{n} P_{n-1+j}, \\
k_Z^*(P_{n-1-j}) &= P_{n-1+j},\ j = 0, \ldots, n-2,\ k_Z^*(P_{n-1+j}) = P_{n-1-j},\ j = 0, \ldots, n-2, \\
k_Z^*(P_{n-1+n-1}) &= H_Z - E_1 - Q - \sum_{j=1}^{n-1}(j+1)P_j - n\sum_{j=1}^{n} P_{n-1+j} - E_2, \\
k_Z^*(P_{n-1+n}) &= E_2 + C_2 = H_Z,\ k_Z^*(E_2) = P_{n-1+n}.
\end{aligned}$$

The spectral radius of k_Z^* can be computed as the greatest real zero of the characteristic polynomial of the matrix representation M_1 of k_Z^* restricted to $\{H_Z, Q, P_{n-1+n-1}, P_{n-1+n}, E_2\}$, which is

$$M_1 = \begin{pmatrix}
2n+1 & -(n+1) & -(n^2+n-2) & -(n^2+n-2) & -n \\
1 & -1 & -(n-1) & -(n-1) & 0 \\
1 & -1 & -n & -n & -1 \\
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{pmatrix}$$

. The characteristic polynomial $P(x) = det(M_1 - xI)$ of M_1 is

$$P(x) = -(x-1)(x+1)(x^3 - nx^2 - 2x - 1).$$

From this the conclusions of Lemma 9 follow. $\square$

4. Case $n = deg(F)$ is odd; $a_0 = 2/(m+1)$ for some $m = 0, 1, 2, \ldots$; $a_0 \neq \frac{n+1}{2} + \frac{l}{2(l+1)}$ for any $l = 0, 1, 2, \ldots$; $L_i(a_0, a_1, \ldots, a_n) = 0$ for any $0 \leq i \leq n-2$.

Lemma 10. *The spectral radius of $k_Z^* : Pic(Z) \rightarrow Pic(Z)$ is the largest real root of the polynomial $x^{2m}(x^3 - nx^2 - 2x - 1) + x^2 + x + n$.*

Proof. Let Z be the space constructed in Theorem 4. Let $H_Z, E_1, Q, P_1, \ldots, P_{n-1+n}, E_2, R_0, R_1, \ldots, R_m, Q_1, \ldots, Q_m$ be a basis for $Pic(Z)$. Then $k_Z^* : Pic(Z) \rightarrow Pic(Z)$

acts as

$$\begin{aligned}
k_Z^*(H_Z) &= (2n+1)H_Z - nE_1 - (n+1)Q - (n+1)\sum_{j=1}^{n-1} jP_j - \sum_{j=1}^{n-1}(n^2-1+j)P_{n-1+j} \\
&\quad -(n^2+n-2)P_{n-1+n} - nE_2 - (n+1)\sum_{j=1}^{m} Q_j - (n+1)R_m, \\
k_Z^*(E_1) &= E_1,\ k_Z^*(Q) = H_Z - E_1 - Q - \sum_{j=1}^{n-1} jP_j - (n-1)\sum_{j=1}^{n} P_{n-1+j} - \sum_{j=1}^{m} Q_j - \sum_{j=0}^{m} R_j, \\
k_Z^*(P_{n-1-j}) &= P_{n-1+j},\ j=0,\dots,n-2,\ k_Z^*(P_{n-1+j}) = P_{n-1-j},\ j=0,\dots,n-2, \\
k_Z^*(P_{n-1+n-1}) &= H_Z - E_1 - Q - \sum_{j=1}^{n-1}(j+1)P_j - n\sum_{j=1}^{n} P_{n-1+j} - E_2 - \sum_{j=1}^{m} Q_j, \\
k_Z^*(P_{n-1+n}) &= E_2 + C_2 = H_Z - R_m,\ k_Z^*(E_2) = P_{n-1+n}, \\
k_Z^*(R_0) &= C_4 = 2H_Z - E_1 - 2Q - \sum_{j=1}^{n-1} jP_j - (n-1)\sum_{j=1}^{n} P_{n-1+j} \\
&\quad -E_2 - 2\sum_{j=1}^{m} Q_j - R_m, \\
k_Z^*(R_j) &= Q_j,\ j=1,2,\dots,m,\ k_Z^*(Q_j) = R_{j-1},\ j=1,2,\dots,m.
\end{aligned}$$

The spectral radius of k_Z^* can be computed as the greatest real zero of the characteristic polynomial of the matrix representation M_1 of k_Z^* restricted to $\{H_Z, Q, P_{n-1+n-1}, P_{n-1+n}, E_2, R_0, \dots, R_m, Q_1, \dots, Q_m\}$, which is

$$P(x) = -(x-1)x(x+1)[x^{2m}(x^3 - nx^2 - 2x - 1) + x^2 + x + n].$$

From this the conclusions of Lemma 10 follow. $\square$

5. Case $n = deg(F)$ is odd; $a_0 \neq 2/(m+1)$ for any $m = 0,1,2,\dots$; $a_0 = \frac{n+1}{2} + \frac{l}{2(l+1)}$ for some $l = 0,1,2,\dots$; $L_i(a_0, a_1, \dots, a_n) = 0$ for any $0 \le i \le n-2$.

Lemma 11. *The spectral radius of $k_Z^* : Pic(Z) \to Pic(Z)$ is the largest real root of the polynomial $x^{2l+2}(x^3 - nx^2 - 2x - 1) + nx^2 + x + 1$.*

Proof. Let Z be the space constructed in Subcase 2.2 of Lemma 6. Denote

$$\begin{aligned}
s_0 &= k_{Y_n}(ep_n) \in E_2, \\
t_1 &= k_{Y_n}^2(ep_n) \in P_{n-1+n}, \\
\dots \\
t_l &= k_{Y_n}^{2l}(ep_n) \in P_{n-1+n},\ s_l = k_{Y_n}^{2l+1}(ep_n) = 0 \in E_2.
\end{aligned}$$

Let $P_{n-1+n+1}$ be the exceptional fiber of blowup at ep_n, S_j the exceptional fiber of blowup at s_j, and T_j the exceptional fiber of blowup at t_j. Let $H_Z, E_1, Q, P_1, \dots, P_{n-1+n}, P_{n-1+n+1}, E_2,$

$S_0, S_1, \ldots, S_m, T_1, \ldots, T_m$ be a basis for $Pic(Z)$. In this basis then

$$\begin{aligned}
C_1 &= H_Z - E_1 - Q - \sum_{j=1}^{n-1}(j+1)P_j - n\sum_{j=1}^{n+1}P_{n-1+j} - E_2 - n\sum_{j=1}^{l}T_j - \sum_{j=0}^{l}S_j,\\
C_2 &= H_Z - E_1 - Q - E_2 - \sum_{j=0}^{m}S_j,\\
C_3 &= H_Z - E_1 - Q - \sum_{j=1}^{n-1}jP_j - (n-1)\sum_{j=1}^{n+1}P_{n-1+j} - (n-1)\sum_{j=1}^{l}T_j,\\
C_4 &= 2H_Z - E_1 - 2Q - \sum_{j=1}^{n-1}jP_j - (n-1)\sum_{j=1}^{n+1}P_{n-1+j} - E_2 - (n-1)\sum_{j=1}^{l}T_j - \sum_{j=0}^{l-1}S_j - 2S_m.
\end{aligned}$$

To justify these formulas note that in the local coordinate for E_2 chosen in Lemma 6

$$C_1 \cap E_2 = [\infty]_{E_2},\ C_2 \cap E_2 = [1]_{E_2},\ C_3 \cap E_2 = \emptyset,\ C_4 \cap E_2 = [0]_{E_2}.$$

Then from the condition imposed on a_0, it follows that $s_0, \ldots, s_m \notin C_1 \cup C_2 \cup C_3$, while C_4 goes through $s_m = [0]_{E_2}$ with multiplicity 1, and $s_0, \ldots, s_{m-1} \notin C_4$. Then $k_Z^* : Pic(Z) \to Pic(Z)$ acts as

$$\begin{aligned}
k_Z^*(H_Z) &= (2n+1)H_Z - nE_1 - (n+1)Q - (n+1)\sum_{j=1}^{n-1}jP_j\\
&\quad -\sum_{j=1}^{n-1}(n^2-1+j)P_{n-1+j} - (n^2+n-2)P_{n-1+n} - nE_2\\
&\quad -(n^2+n-2)\sum_{j=1}^{l}T_j - n\sum_{j=0}^{l-1}S_j - 2nS_l,\\
k_Z^*(E_1) &= E_1,\ k_Z^*(Q) = H_Z - E_1 - Q - \sum_{j=1}^{n-1}jP_j - (n-1)\sum_{j=1}^{n}P_{n-1+j} - (n-1)\sum_{j=1}^{l}T_j,\\
k_Z^*(P_{n-1-j}) &= P_{n-1+j},\ j = 0, \ldots, n-2,\ k_Z^*(P_{n-1+j}) = P_{n-1-j},\ j = 0, \ldots, n-2,\\
k_Z^*(P_{n-1+n-1}) &= H_Z - E_1 - Q - \sum_{j=1}^{n-1}(j+1)P_j - n\sum_{j=1}^{n}P_{n-1+j} - E_2 - \sum_{j=0}^{l}S_j,\\
k_Z^*(P_{n-1+n}) &= E_2,\ k_Z^*(E_2) = P_{n-1+n},\\
k_Z^*(P_{n-1+n+1}) &= H_Z - E_2 - \sum_{j=0}^{l}S_j,\\
k_Z^*(S_0) &= P_{n-1+n+1},\ k_Z^*(S_j) = T_j,\ j = 1, \ldots, l,\ k_Z^*(T_j) = S_{j-1},\ j = 1, \ldots, l.
\end{aligned}$$

The spectral radius of k_Z^* can be computed as the greatest real zero of the characteristic polynomial of the matrix representation M_1 of k_Z^* restricted to $\{H_Z, Q, P_{n-1+n-1}, P_{n-1+n+1}, S_0, \ldots, S_l, T_1, \ldots, T_l\}$. The characteristic polynomial $P(x) = det(M_1 - xI)$ of M_1 is

$$P(x) = -[x^{2l+2}(x^3 - nx^2 - 2x - 1) + nx^2 + x + 1].$$

From this the conclusions of Lemma 11 follow. □

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